LESSON

17 The Fundamental Theorem of Calculus

Here comes the resounding climax of calculus. It would be best to read this lesson with some bombastic orchestral music like that of Wagner or Orff. This, however, is not necessary. The initial question here is innocent enough: If we make a function from that "area under a curve" stuff, what would its derivative be? So suppose that our curve is $y = f(t)$ (see Figure 17.1). We use the variable t in order to save x for something more important.

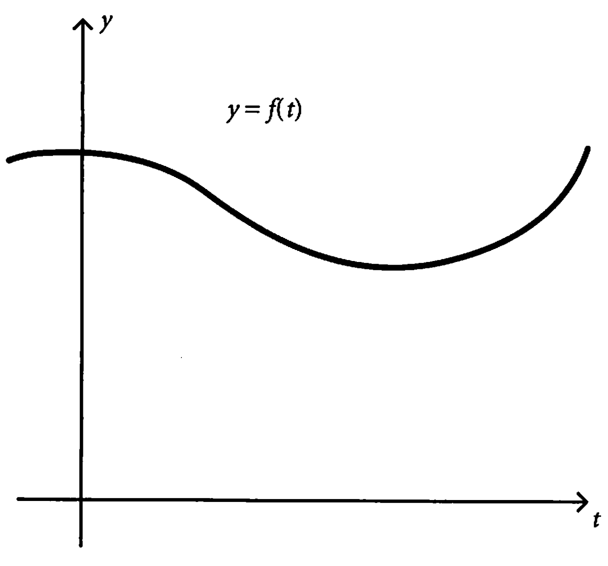

Figure 17.1

Now let our "area under the curve function" be $g(x)$ = the area under the curve $y = f(t)$ between 0 and some point x. Therefore, $g(x) = \int_0^x f(t)\,dt$. This area is illustrated in Figure 17.2.

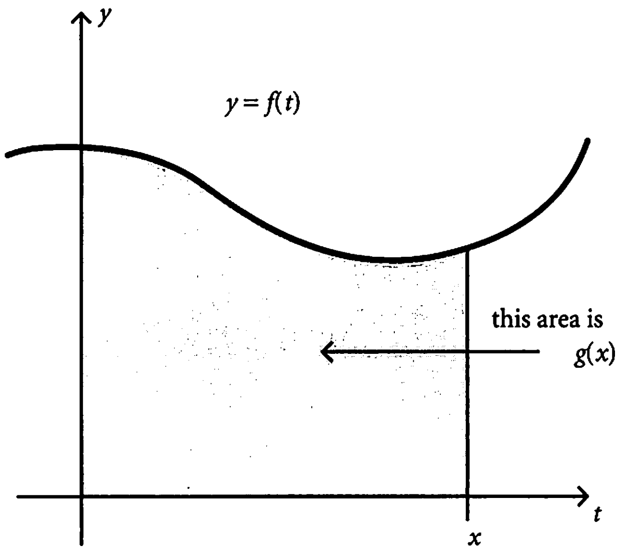

Figure 17.2

Example

If $f(t) = 2t$ and $g(x) = \int_0^x f(t)\,dt$, then what is $g(3)$?

Solution

$g(3) = \int_0^3 f(t)\,dt = \int_0^3 2t\,dt$ = the area beneath the curve $y = 2t$ from 0 to 3. The graph of $f(t) = 2t$ is shown in Figure 17.3. This area is a triangle with base 3 and height 6, so $g(3) = \int_0^3 2t\,dt = \frac{1}{2}(3)(6) = 9$.

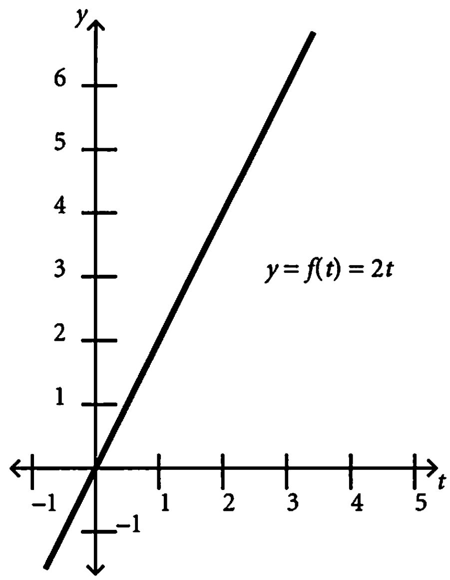

Figure 17.3

▶ Practice

Suppose $f(t) = \frac{1}{2}t + 1$ and $g(x) = \int_0^x f(t)\,dt$. Evaluate the following.

1. $g(1)$

2. $g(2)$

3. $g(3)$

4. $g(4)$

5. $g(5)$

6. $g(0)$

Now suppose $f(t) = 7$ and $g(x) = \int_0^x f(t)\,dt$. Evaluate the following.

7. $g(0)$

8. $g(1)$

9. $g(2)$

10. $g(3)$

11. $g(4)$

12. $g(5)$

Now that the concept of $g(x) = \int_0^x f(t)\,dt$ is clear, we can answer the next question: What is the derivative of $g(x)$?

Begin with the definition of the derivative.

$$g'(x) = \lim_{h\to 0}\frac{g(x+h) - g(x)}{h}$$

Use $g(x) = \int_0^x f(t)\,dt$.

$$g'(x) = \lim_{h\to 0}\frac{\int_0^{x+h} f(t)\,dt - \int_0^x f(t)\,dt}{h}$$

Use $\int_a^c f(t)\,dt - \int_a^b f(t)\,dt = \int_b^c f(t)\,dt$.

$$g'(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$

Now the integral $\int_x^{x+h} f(t)\,dt$ represents the skinny little area just to the right of point x (see Figure 17.4). This is *almost* a rectangle with a base of h and a height of $f(x)$, so the integral $\int_x^{x+h} f(t)\,dt$ is *almost* $h \cdot f(x)$. As h gets really small, this area gets closer to being a rectangle (see Figure 17.5).

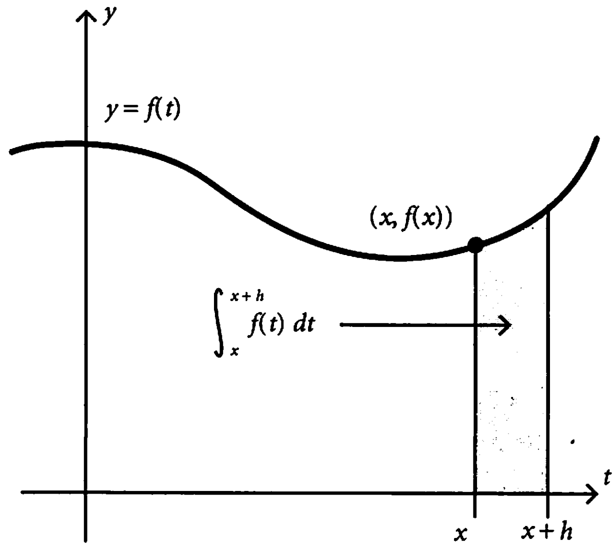

Figure 17.4

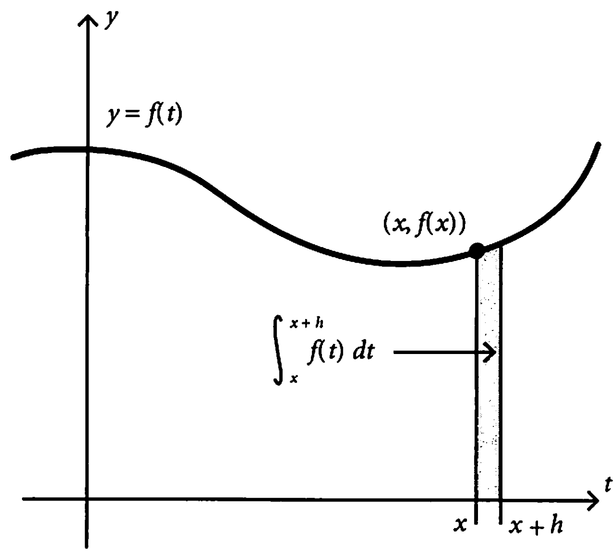

Figure 17.5

Therefore, as h approaches zero, this integral approaches $h \cdot f(x)$, thus:

$$g'(x) = \lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}$$

$$= \lim_{h\to 0}\frac{h \cdot f(x)}{h} = \lim_{h\to 0} f(x) = f(x)$$

The Fundamental Theorem of Calculus

The Fundamental Theorem of Calculus can be written as follows:

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx = g(b) - g(a)\text{, where } g'(x) = f(x)$$

What does this mean? It means that the derivative of the function $g(x)$, which represents "the area under the curve," is the very function $f(x)$ used to draw the curve. It came as an amazing surprise to the world of mathematics that the process of finding the slope of a tangent line and the process of finding the area under a curve were exact opposites. In order to find the area under a curve $y = f(x)$, we need to find a function $g(x)$ whose derivative is $f(x)$.

We can use this to evaluate $\int_a^b f(x)\,dx$ by the Fundamental Theorem of Calculus (see Figure 17.6).

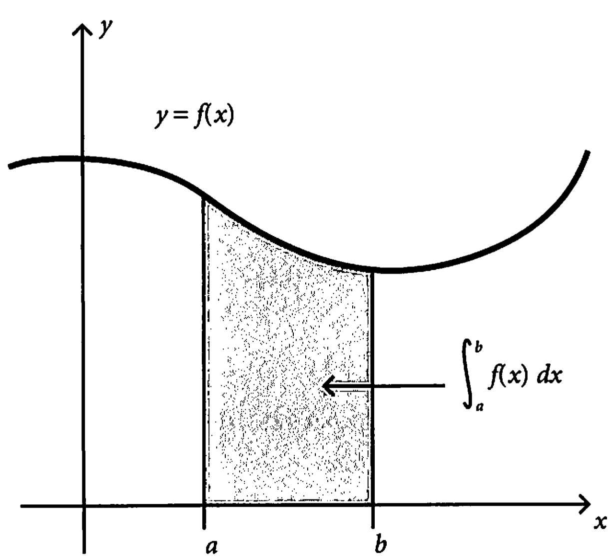

Figure 17.6

For example, the derivative of $g(x) = x^2$ is $g'(x) = 2x$. Thus, the area under $f(x) = 2x$ between $x = 3$ and $x = 5$ is $\int_3^5 2x dx = g(5) - g(3) = 5^2 - 3^2 = 16$. This is exactly the area of the trapezoid under the line $y = 2x$ between $x = 3$ and $x = 5$. However, here the Fundamental Theorem of Calculus saves us from having to draw out the graph of $y = 2x$.

Example

The derivative of $g(x) = \frac{1}{3}x^3$ is $g'(x) = x^2$. Use this to evaluate $\int_{-1}^{2} x^2\,dx$.

Solution

By the Fundamental Theorem of Calculus, $\int_a^b f(x)\,dx = g(b) - g(a)$, where $g'(x) = f(x)$. Thus,

$$\int_{-1}^{2} x^2\,dx = g(2) - g(-1)$$

$$= \frac{1}{3}(2)^3 - \frac{1}{3}(-1)^3 = \frac{8}{3} + \frac{1}{3} = 3.$$

Even if we *had* drawn out the graph of $y = x^2$, how would we have been able to guess that the area of the two shaded curves add up to exactly three? This is why the Fundamental Theorem of Calculus is so powerful! (See Figure 17.7.)

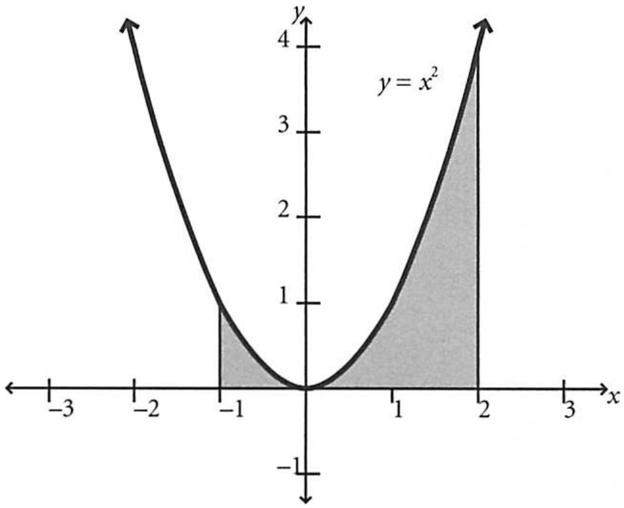

Figure 17.7

Example

If $g(x) = x^4$, then $g'(x) = 4x^3$. Use this to evaluate $\int_{-1}^{1} 4x^3\,dx$.

Solution

$$\int_{-1}^{1} 4x^3\,dx = g(1) - g(-1)$$
$$= 1^4 - (-1)^4 = 1 - 1 = 0$$

The answer is zero because there is exactly as much area above the x-axis (which counts positively) as there is below the x-axis (which counts negatively).

▶ Practice

If $g(x) = x^2 + x$ then $g'(x) = 2x + 1$. Use this to evaluate the following.

13. $\int_{1}^{3} (2x + 1)\,dx$

14. $\int_{-3}^{1} (2x + 1)\,dx$

15. $\int_{2}^{6} (2x + 1)\,dx$

16. $\int_{0}^{4} (2x + 1)\,dx$

Use $\frac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}}\right) = \sqrt{x}$ to evaluate the following.

17. $\int_{0}^{1} \sqrt{x}\,dx$

18. $\int_{0}^{4} \sqrt{x}\,dx$

19. $\int_{4}^{9} \sqrt{x}\,dx$

20. $\int_{0}^{100} \sqrt{x}\,dx$

Use $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$ to evaluate the following.

21. $\int_{1}^{2} \frac{1}{x^2}\,dx$

22. $\int_{1}^{5} \frac{1}{x^2}\,dx$

23. $\int_{2}^{4} \frac{1}{x^2}\,dx$

24. $\int_{-3}^{-1} \frac{1}{x^2}\,dx$